The Dimension of Spline Spaces with Highest Order Smoothness over Hierarchical T-meshes

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Abstract

This paper discusses the dimension of spline spaces $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ over certain type of hierarchical T-meshes. The major step is to set up a bijection between the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ and a univariate spline space whose definition depends on the l-edges of the extended T-mesh. We decompose the univariate spline space into direct sums in the sense of isomorphism using the theory of the short exact sequence in homological algebra. According to the decomposition of the univariate spline space, the dimension formula of the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ over certain type of hierarchical T-mesh is presented. A set of basis functions of the spline space is also constructed.

Keywords: Dimension formula, spline space, T-mesh, homology.

1 Introduction

Non-Uniform Rational B-Splines (NURBS) are popular tool to represent surface models in Computer Aided Geometric Design (CAGD) and Computer Graphics. However, due to the tensor-product structure of NURBS, local refinement of surface models based on NURBS is impossible, and NURBS models generally contain large number of superfluous control points. To overcome the above drawbacks, Sederberg et al. introduced T-splines, the control meshes of which allow T-junctions ([1, 2]). T-splines provide local refinement strategy and can reduce the large number of superfluous control points in NURBS models.

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In [3], the concept of spline spaces over T-meshes is introduced. Different from T-splines, a spline over a T-mesh is a single polynomial within each cell of the T-mesh, and it achieves the specified smoothness across the common edges. Spline spaces over T-meshes are suitable for geometric modeling[4] and they can be applied for analysis naturally, since it is easy to do standard Finite Element Method analysis based on splines over T-meshes [5].

One major issue in the theory of splines over T-meshes is to study the dimension of the spline spaces. There have been a few literature focusing on the problem so far. In [3], the dimension formula for the spline space $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ is obtained with constraints $m \geq 2\alpha+1, n \geq 2\beta+1$. The result is further improved by Li et al. [6]. For the spline spaces with highest order smoothness where $m \leq 2\alpha, n \leq 2\beta$, the authors of the current paper derived a dimension formula for C^1 biquadratic spline spaces (that is, $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$) over hierarchical T-meshes[7]. Recently, B. Mourrain gives a general formula for the spline spaces $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ by homological techniques[8]. Unfortunately there is a term in the dimension formula which is very hard to compute in practice.

Contrary to the above positive results, Li and Chen have showed that the dimension of spline spaces with highest order smoothness over T-meshes may depend on the geometry besides the topology information of the T-meshes [9]. The result suggests that it is vain to study the dimension formula over general T-meshes in the case of highest order of smoothness. In this paper, the dimension formula of spline spaces $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ over certain type of hierarchical T-meshes will be explored.

There are several methods for establishing the dimension of spline spaces, such as the B-net method [3], and the smoothing cofactor method [10] and homology method [11]. In the smoothing cofactor method, the cofactor of a spline associated with the common edge of two adjacent cells is a univariate polynomial. This is similar to homology theory in topology. Therefore, using the smoothing cofactor method and according to a result in [7], we construct an isomorphism between a spline space with highest order smoothness over a T-mesh and a univariate spline space satisfying some conditions. For a certain type of hierarchical T-mesh, by giving an order of the interior l-edges of the extended T-mesh, we can then decompose the univariate spline space into a direct sum, and thus give a dimension formula for the spline space over hierarchial T-meshes.

The rest of the paper is organized as follows. In Section 2, the definitions and some results regarding T-meshes and spline spaces over T-meshes are reviewed. In Section 3, an equivalence is set up between the spline over a T-mesh and a univariate spline space. The proof of the dimension formula is presented in Section 4. Some examples are also provided. In Section 5, we conclude the paper with future research problems. We leave the proof for a key lemma and the construction of a set of basis functions of the spline space in the appendix.

2 Spline Spaces over T-meshes

In this section, we first review some concepts about T-meshes and spline spaces over T-meshes.

2.1 Spline spaces over T-meshes

A **T-mesh** is a rectangular grid that allows T-junctions. For simplicity, in this paper we consider only regular T-meshes whose boundary grid lines form a rectangle. We adopt the same definitions for **vertex**, **edge**, and **cell** as in [3], and the definitions for **l-edge**, **interior l-edge**, **associated tensor product mesh** are borrowed from [7]. A grid point in a T-mesh is called a vertex of the T-mesh. Vertices of a T-mesh are divided into different types. For example, in Figure 1, $\{b_i\}_{i=1}^{i=10}$ are boundary vertices and $\{v_i\}_{i=1}^{i=5}$ are interior vertices. v_2 is a crossing vertex and $\{b_i\}_{i=1}^{i=10} \cup \{v_i\}_{i=1}^{i=5} - \{v_2, b_1, b_3, b_6, b_8\}$ are T-vertices. The line segment connecting two adjacent vertices on a grid line is called an edge of T-mesh such as $v_4v_5, b_9b_{10}, v_2v_3$ in Figure 1. b_2v_3 is a large edge (l-edge for short), which is the longest possible line segment consisting of several edges. The boundary of a regular T-mesh consists of four l-edges,which are called boundary l-edges. The other l-edges in T-mesh are called interior l-edges. A regular T-mesh can be extended to a tensor product mesh, called the associated tensor-product mesh, by extending all the interior large edges to the boundary, see Figure 1.

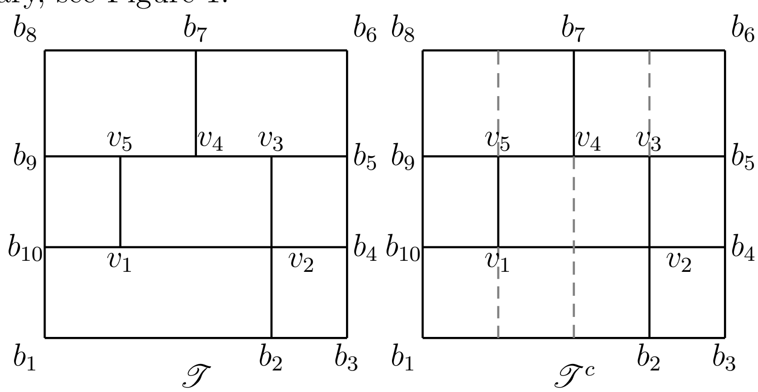


Figure 1: A T-mesh $\mathscr{T}$ and its associated tensor product mesh $\mathscr{T}^c$.

Given a T-mesh $\mathscr{T}$, $\mathscr{F}$ is the set of all the cells of $\mathscr{T}$ and Ω is the region occupied by cells in $\mathscr{F}$. Spline spaces over T-meshes are defined by

$$\mathbf{S}(m, n, \alpha, \beta, \mathscr{T}) := \{f(x, y) \in C^{\alpha,\beta}(\Omega) : f(x, y)|\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F}\}, \qquad (1)$$

where $\mathbb{P}_{mn}$ is the space of all the polynomials with bi-degree (m, n), and $C^{\alpha,\beta}$ is the space consisting of all the bivariate functions that are continuous in Ω with order α along the x direction and order β along the y direction. In this paper, we will focus on the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$.

2.2 Hierarchical T-meshes and extended T-meshes

A hierarchical T-mesh is a special type of T-mesh that has a natural level structure [4]. It is defined in a recursive fashion. Initially a tensor product mesh (level 0) is presumed. From level k to $k+1$, a cell is subdivided at level k into four sub-cells, which are cells at level $k+1$, by connecting the middle points of the opposite edges with two straight lines. Figure 2 illustrates a sequence of hierarchical T-meshes.

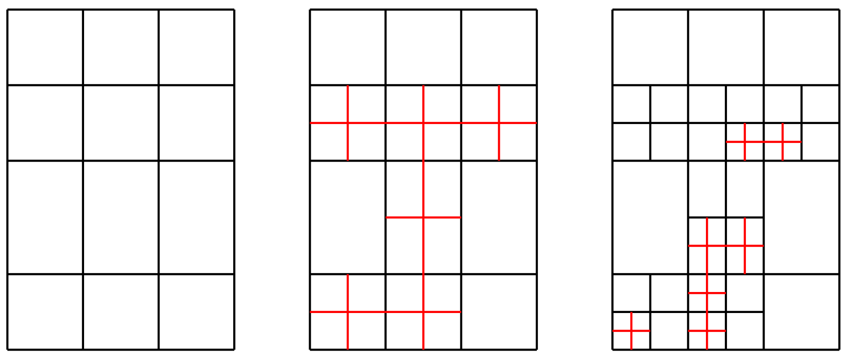


Figure 2: Hierarchical T-meshes.

For a T-mesh $\mathscr{T}$, the extended T-mesh $\mathscr{T}^{\varepsilon}$ associated with $\mathscr{T}$ is an enlarged T-mesh by copying each horizontal boundary line of $\mathscr{T}$ m times, and each vertical boundary line of $\mathscr{T}$ n times, and by extending all the line segments with an end point on the boundary of $\mathscr{T}$ [7]. This can be made precise by the following example. Figure 3 illustrates a T-mesh (left) and the extended T-mesh (right) associated with degree $(3,3)$.

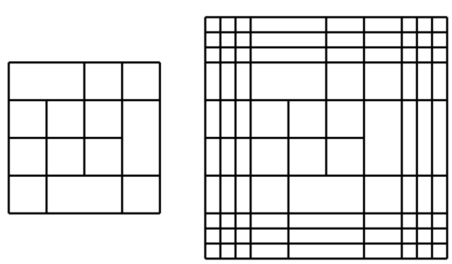


Figure 3: A T-mesh $\mathscr{T}$ and its extended T-mesh associated with degree $(3,3)$.

2.3 Homogeneous boundary conditions

A spline space over a given T-mesh $\mathscr{T}$ with homogeneous boundary conditions is defined by [7]

$$\overline{\mathbf{S}}(m,n,\alpha,\beta,\mathscr{T}) := \{f(x,y) \in C^{\alpha,\beta}(\mathbb{R}^2) : f(x,y)|_\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F} \text{ and } f|_{\mathbb{R}^2\setminus\Omega} \equiv 0\},$$

where $\mathbb{P}_{mn}, \mathscr{F}$ is defined as before. One important observation in [7] is that the two spline spaces $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$ and $\overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^\varepsilon)$ are closely related.

Theorem 2.1. *[7] Given a T-mesh $\mathscr{T}$, and let $\mathscr{T}^\varepsilon$ be the extended T-mesh associated with $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$. Then*

$$\mathbf{S}(m,n,m-1,n-1,\mathscr{T}) = \overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^\varepsilon)|_{\mathscr{T}}, \tag{2}$$

$$\dim \mathbf{S}(m,n,m-1,n-1,\mathscr{T}) = \dim \overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^\varepsilon). \tag{3}$$

Based on the above theorem, we only have to consider spline spaces over T-meshes with homogeneous boundary conditions.

3 Equivalent spline spaces

In this section, we will give an equivalent description of spline spaces $\overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^\varepsilon)$ by using the smoothing cofactor method.

3.1 Smoothing cofactor method

As before, let $\mathscr{T}$ and $\mathscr{T}^\varepsilon$ be a T-mesh and its extension respectively. Let $\mathscr{F} = \{C_1, C_2, \cdots, C_k\}$ be the set of all the cells in $\mathscr{T}^\varepsilon$, and Ω_i be the region occupied by $C_i \in \mathscr{F}$. Denote $\Omega_{k+1} = \mathbb{R}^2 \setminus \cup_{i=1}^k \Omega_i$ and $\mathscr{F}^\varepsilon = \{\Omega_1, \Omega_2, \cdots, \Omega_{k+1}\}$.

Let $U_1, U_2, U_3, U_4 \in \mathscr{F}^\varepsilon$ be four regions in adjacent positions as shown in Figure 4. For a spline function $f(x,y) \in \overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^\varepsilon)$, denote $f_i(x,y)$ be the bivariate polynomial which coincides with $f(x,y)$ on U_i, $i = 1,2,3,4$. Note that if $f_i(x,y) = f_j(x,y)$ for two adjacent cells U_i and U_j, then U_i and U_j can be merged into a single region, and in this case (x_0,y_0) is a T-junction.

By the smoothing cofactor method, we have the following relationship between $f_i(x,y)$:

Lemma 3.1. *[6, 9] Let $f_i(x,y), i = 1,2,3,4$ be defined as in the preceding paragraph. Then there exist a constant $k \in \mathbb{R}$ and polynomials $a(y) \in \mathbb{P}_n[y]$, $b(x) \in \mathbb{P}_m[x]$ such*

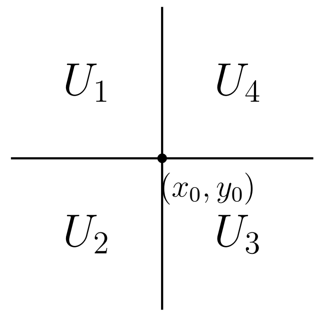


Figure 4: Smoothing conditions in adjacent cells

that

$$\begin{aligned}
f_1(x,y) &= f_2(x,y) + b(x)(y-y_0)^n, \\
f_3(x,y) &= f_2(x,y) + a(y)(x-x_0)^m, \\
f_4(x,y) &= f_2(x,y) + a(y)(x-x_0)^m + b(x)(y-y_0)^n + k(x-x_0)^m(y-y_0)^n,
\end{aligned}$$

where (x_0, y_0) *is the vertex of* $\mathscr{T}^\varepsilon$ *in Figure 4. Furthermore,* $a(y), b(x)$ *and* k *are uniquely determined by* $f_i(x,y)$*,* $i = 1,2,3,4$*, specifically*

$$a(y) = \frac{1}{m!}\left(\frac{\partial^m f_3(x_0,y)}{\partial x^m} - \frac{\partial^m f_2(x_0,y)}{\partial x^m}\right), \tag{4}$$

$$b(x) = \frac{1}{n!}\left(\frac{\partial^n f_1(x,y_0)}{\partial y^n} - \frac{\partial^n f_2(x,y_0)}{\partial y^n}\right), \tag{5}$$

and

$$k = \frac{1}{m!}\frac{1}{n!}\frac{\partial^{m+n}(f_2(x,y) + f_4(x,y) - f_1(x,y) - f_3(x,y))}{\partial x^m \partial y^n}. \tag{6}$$

$a(y)$ *and* $b(x)$ *are smoothing co-factors associated the edges between adjacent cells. The constant* k *is called the conformality factor associated with the common vertex* (x_0, y_0)*.*

Based on the above lemma, we convert the dimension problem into the study of a univariate spline space. We first introduce the following definition.

Definition 3.1. *Let* E^h *be a horizontal l-edge in* $\mathscr{T}^\varepsilon$*, and* $\{v_1, v_2, \cdots, v_r\}$ *be vertices on* E^h*. Assume the x-coordinates of the vertices are* $x_1 < x_2 < \ldots < x_r$ *respectively. We define a univariate spline space with homogeneous boundary conditions associated with* E^h*:*

$$\begin{aligned}
\overline{\mathbf{S}}(m, m-1, E^h) := \big\{ & p(x) \in C^{m-1}(\mathbb{R}) : p(x)|_{[x_i, x_{i+1}]} = p_i(x) \in \mathbb{P}_m[x], \\
& i = 1, 2, \ldots, r-1, \ \ \textit{and} \ \ p(x)|_{\mathbb{R}\setminus[x_1, x_r]} \equiv 0 \big\}.
\end{aligned} \tag{7}$$

Similarly, for a vertical l-edge E^v, we can define a univariate spline space $\overline{\mathbf{S}}(n, n-1, E^v)$ associated with it.

By the homogeneous boundary conditions, it is easy to see that for any $p(x) \in \overline{\mathbf{S}}(m, m-1, E^h)$, there exist constants $k_1, \ldots, k_r$ such that

$$\sum_{i=1}^{r} k_i (x - x_i)^m \equiv 0. \tag{8}$$

Equation (8) is equivalent to a linear system associated with E^h:

$$\begin{cases} \sum_{i=1}^{r} k_i = 0, \\ \sum_{i=1}^{r} k_i x_i = 0, \\ \cdots, \\ \sum_{i=1}^{r} k_i x_i^m = 0. \end{cases} \tag{9}$$

For a vertical l-edge E^v, there is a similar equation

$$\sum_{i=1}^{s} k_i (y - y_i)^n \equiv 0, \tag{10}$$

which is also equivalent to a linear system associated with E^v:

$$\begin{cases} \sum_{i=1}^{s} k_i = 0, \\ \sum_{i=1}^{s} k_i y_i = 0, \\ \cdots, \\ \sum_{i=1}^{s} k_i y_i^n = 0. \end{cases} \tag{11}$$

By (9) and (11), one immediately has

Lemma 3.2. *[6]*

$$\dim \overline{\mathbf{S}}(m, m-1, E^h) = (r - m - 1)_+, \qquad \dim \overline{\mathbf{S}}(n, n-1, E^v) = (s - n - 1)_+.$$

Here $u_+ = \max(0, u)$.

3.2 Conformality vector spaces

To study the dimension of spline space $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$, we need to consider the conformality conditions for all the (horizontal and vertical) l-edges. Thus we introduce the following definition.

Definition 3.2. *Let E_i^h, $i = 1, 2, \ldots, p$ be all the horizontal l-edges of $\mathscr{T}^\varepsilon$, and E_j^v, $j = 1, 2, \ldots, q$ be all the vertical l-edges of $\mathscr{T}^\varepsilon$. Define a linear space $W[\mathscr{T}^\varepsilon]$ by*

$$W[\mathscr{T}^\varepsilon] := \{\mathbf{k} = (k_1, k_2, \cdots, k_v)^T : L_i^h = 0, L_j^v = 0,\ i = 1, \cdots, p,\ j = 1, \cdots, q\},$$

where v is the number vertices of $\mathscr{T}^\varepsilon$, k_i is the conformality factor corresponding to i-th vertex of $\mathscr{T}^\varepsilon$, and $L_i^h = 0$ and $L_j^v = 0$ are linear systems associated with the l-edges E_i^h and E_j^v respectively. $W[\mathscr{T}^\varepsilon]$ is called the conformality vector space of $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$. Similarly, one can define the conformality vector space $W[E]$ of $\overline{\mathbf{S}}(m, m-1, E^h)$ (or $\overline{\mathbf{S}}(n, n-1, E^v)$) associated with a l-edge.

The following facts should be noted regarding $W[\mathscr{T}^\varepsilon]$.

1. $W[\mathscr{T}]$ can also be defined over a general T-mesh $\mathscr{T}$ (besides an extended T-mesh). For any $f(x, y) \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, there is a unique corresponding vector $\mathbf{k} \in W[\mathscr{T}]$ called **conformality vector**.

2. k_i, which is the $i-$th component of $\mathbf{k}$, is the conformality factor corresponding to $i-$th vertex in $\mathscr{T}^\varepsilon$. This vertex is the intersection point of two l-edges. So k_i has to satisfy both equations (9) and (11) associated with the two l-edges. Once $\mathbf{k}$ is determined, the smoothing co-factors $a(y)$ and $b(x)$ can be constructed accordingly. Figure 5 illustrates an example, where $a(y) = k_5(y - y_1)^n + k_4(y - y_2)^n + k_3(y - y_3)^n$, $b(x) = k_1(x - x_1)^m + k_2(x - x_2)^m$. $a(y)$ is determined by k_3, k_4, k_5 whose corresponding vertices lie beneath the edge associating with $a(y)$; $b(x)$ is determined by k_1, k_2 whose corresponding vertices lie on the left of the edge corresponding to $b(x)$.

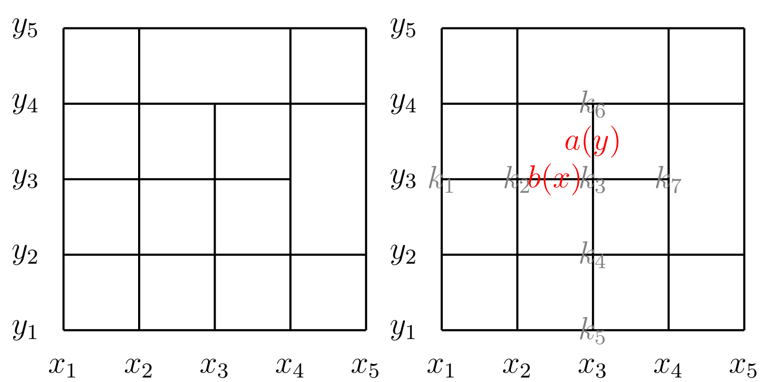


Figure 5: Conformality vector

3. A naught value of k_i which is the conformality factor corresponding to a T-vertex results in the vanishing of the corresponding vertex in the T-mesh $\mathscr{T}$. An example is illustrated in Figure 6 where $k_7 = 0$, and in this case $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}) = \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}')$, or $\mathscr{T}$ degenerates to $\mathscr{T}'$.

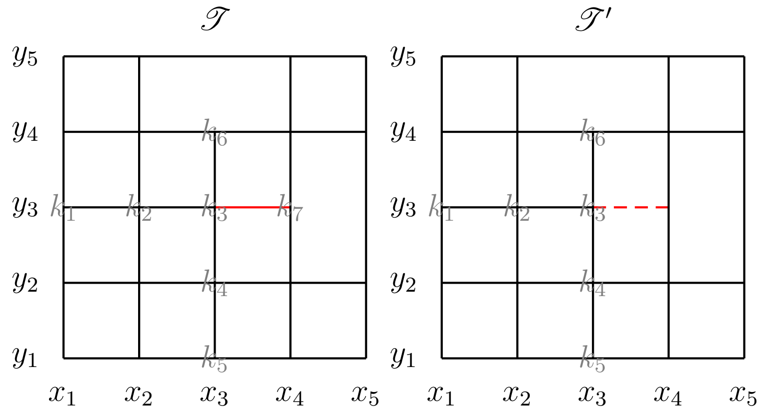


Figure 6: The naught value of a conformality factor.

As an example, we discuss the conformality vectors of B-spline functions.

Example 3.1. *Let $\Delta : -\infty < x_1 < x_2 < \ldots < x_r < \infty$ $(r > m+1)$ be a partition of $\mathbb{R}$, and $\overline{\mathbf{S}}(m, m-1, \Delta)$ be the B-spline function space defined over Δ, that is, $\overline{\mathbf{S}}(m, m-1, \Delta)$ consists of piecewise polynomials of degree m with C^{m-1} continuity over Δ. The B-spine basis function $N[x_i, x_{i+1}, \ldots, x_{i+m+1}]$ is an element in $\overline{\mathbf{S}}(m, m-1, E_i)$, where E_i is the interval $[x_i, x_{i+m+1}]$ with the partition: $x_i < x_{i+1} < \ldots < x_{i+m+1}$. Here $1 \leq i \leq r-m-1$. The conformality vector $\mathbf{k} = (k_i, k_{i+1}, \ldots, k_{i+m+1})$ of $N[x_i, x_{i+1}, \ldots, x_{i+m+1}]$ can be obtained by solving the associated linear system* (9)*. It is easy to see that $k_j k_{j+1} < 0$, $j = i, i+1, \ldots, i+m$ and $k_i > 0$.*

Example 3.2. *Let $\overline{\mathbf{S}}(m, m-1, \Delta_x)$ be the B-spline function space defined over $\Delta_x : -\infty < x_1 < x_2 < \ldots < x_r < \infty$ $(r > m+1)$, and $\overline{\mathbf{S}}(n, n-1, \Delta_y)$ be the B-spline function space defined over $\Delta_y : -\infty < y_1 < y_2 < \ldots < y_s < \infty$ $(s > n+1)$. We denote $\mathscr{T}_\otimes$ as a tensor product mesh $\Delta_x \times \Delta_y$. It is easy to see that, $f(x) \in \overline{\mathbf{S}}(m, m-1, \Delta_x)$ and $g(y) \in \overline{\mathbf{S}}(n, n-1, \Delta_y)$ implies that $f(x)g(y) \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}_\otimes)$.*

Now assume that the B-spline basis functions $N[x_i, x_{i+1}, \ldots, x_{i+m+1}] \in \overline{\mathbf{S}}(m, m-1, \Delta_x)$ and $N[y_j, y_{j+1}, \ldots, y_{j+n+1}] \in \overline{\mathbf{S}}(n, n-1, \Delta_y)$ have conformality vectors $\mathbf{k}^1 := (k^1_i, k^1_{i+1}, \ldots, k^1_{i+m+1})$ and $\mathbf{k}^2 := (k^2_j, k^2_{j+1}, \ldots, k^2_{j+n+1})$ respectively. Then by equation (6)*, the B-spline basis function $N[x_i, x_{i+1}, \ldots, x_{i+m+1}] \cdot N[y_j, y_{j+1}, \ldots, y_{j+n+1}] \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}_\otimes)$ has a conformality vector $\mathbf{k}^1 \otimes \mathbf{k}^2$ which is a vector of dimension $(m+2)(n+2)$ with elements $k^1_p k^2_q$, $p = i, i+1, \ldots, i+m+1$, $q = j, j+1, \ldots, j+n+1$.*

3.3 Equivalence of spline spaces

Based on the above preparations, we obtain a mapping $\mathscr{K}$ between the spline space $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})$ and the conformality vector space $W[\mathscr{T}^{\varepsilon}]$:

$$\mathscr{K} : \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon}) \longrightarrow W[\mathscr{T}^{\varepsilon}]. \tag{12}$$

By Equation (6), $\mathscr{K}$ is a linear mapping due to the linear property of the operator $\partial^{m+n} / \partial x^m \partial y^n$. In fact, $\mathscr{K}$ is an isomorphic mapping.

Theorem 3.3. *The mapping* $\mathscr{K} : \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon}) \longrightarrow W[\mathscr{T}^{\varepsilon}]$ *is bijective.*

***Proof*.** We first prove that $\mathscr{K}$ is injective. It is enough to show that $\mathscr{K} f = \mathbf{0}$ implies $f \equiv 0$ for $f \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})$. By the remark 3 following **Definition** 3.2, if the conformality vector of $f(x, y)$ is a zero vector, then $f(x, y)$ is a single polynomial over $\mathscr{T}^{\varepsilon}$. By the homogeneous boundary conditions, $f(x, y) \equiv 0$. Thus $\mathscr{K}$ is injective.

Next we show $\mathscr{K}$ is surjective. By the remark 2 following **Definition** 3.2, for a given conformality vector $\mathbf{k} \in W[\mathscr{T}^{\varepsilon}]$, one can construct a smoothing cofactor for each edge of the T-mesh $\mathscr{T}^{\varepsilon}$, and thus obtains a spline function $f(x, y) \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})$ corresponding to $\mathbf{k}$, that is, $\mathscr{K}$ is surjective. Thus the mapping $\mathscr{K}$ is bijective. $\square$

By the above theorem, the spline space $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})$ is isomorphic to the conformality vector space $W[\mathscr{T}^{\varepsilon}]$, and

$$\dim \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon}) = \dim W[\mathscr{T}^{\varepsilon}]. \tag{13}$$

Similarly, one can show that

$$\overline{\mathbf{S}}(m, m-1, E^h) \cong W[E^h], \qquad \overline{\mathbf{S}}(n, n-1, E^v) \cong W[E^v] \tag{14}$$

for a horizontal l-edge E^h and a vertical l-edge E^v.

In the following, we only have to analyze the structure of $W[\mathscr{T}^{\varepsilon}]$.

4 The dimension formula

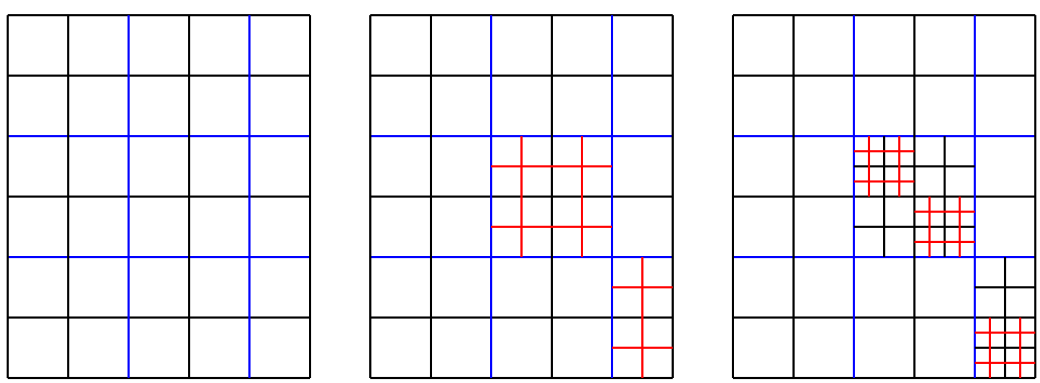
In this section, we will derive a dimension formula for the spline $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})$ over a certain type of hierarchical T-meshes by direct sum decomposition of the spline space. The decomposition is according to each interior l-edge of the T-mesh $\mathscr{T}^{\varepsilon}$. A set of basis functions of the spline space is also constructed.

4.1 Some definitions

We first introduce a certain type of hierarchical T-meshes associated with degree m, n and denote it by $\mathscr{T}_{m,n}$.

Definition 4.1. *Let $\mathscr{T}_{\otimes} = [x_0, x_1, \cdots, x_p] \times [y_0, y_1, \cdots, y_q]$ be a tensor product mesh, and let $X = \{x_0, x_{m-1}, \cdots, x_{s(m-1)}, x_p\}$ and $Y = \{y_0, y_{n-1}, \cdots, y_{t(n-1)}, y_q\}$, where s and t are unique integers such that $0 < p - s(m-1) \leq m-1$, $0 < q - t(n-1) \leq n-1$. The domain $\Omega = [x_0, x_p] \times [y_0, y_q]$ is subdivided by the lines $\{x = x_i, y = y_j : x_i \in X, y_j \in Y\}$ into sub-domains, each of which is occupied by a local tensor product mesh of size $(m-1) \times (n-1)$ (or smaller near the right or upper boundary lines). Each subdomain is called a (m, n)-subdomain of $\mathscr{T}_{\otimes}$. a (m, n)-subdomain is called s boundary one, if it is near the boundary lines of T-mesh. A subdomain is said subdivided if each cell in the subdomain is subdivided (a cell is subdivided by connecting the middle points of oppose sides of the cell with two line segments). A subdomain is called isolated if it is subdivided while its adjacent subdomains (two subdomains are called adjacent if they share a common boundary line segment) are not subdivided and it is not boundary one.*

Figure 7(a) illustrates an example where a tensor product mesh is subdivided into $(3, 3)$-subdomains by blue lines. In Figure 7(b), two subdomains are further subdivided by red lines, and these one subdomain is isolated.



(a) (b) (c)

Figure 7: Hierarchical T-mesh $\mathscr{T}_{m,n}$ in the case of $m = n = 3, p = 5, q = 6$

.

Definition 4.2. *A hierarchical T-mesh $\mathscr{T}_{m,n}$ associated with (m, n) is defined level by level. At level $k = 0$, $\mathscr{T}_{m,n}$ is a tensor product mesh $\mathscr{T}_{\otimes}$. From level 0 to level 1,*

subdivide $\mathscr{T}_\otimes$ into (m,n)-subdomains, and some of which are further subdivided to get the mesh at level 1. Generally from level k to level $k+1$, subdivide some local tensor product meshes at level k (which are obtained by subdividing subdomains at level $k-1$) into (m,n)-subdomains at level k, some of which are further subdivided to get level $k+1$ mesh.

The T-mesh $\mathscr{T}_{m,n}$ is a special type of hierarchical T-mesh [4]. In particular, it is a regular hierarchical T-mesh when $m = n = 2$. Figure 7 illustrates the process of generating a hierarchical T-mesh $\mathscr{T}_{3,3}$. Figure 7(a) shows that a tensor product mesh is subdivided into subdomains at level $k = 0$. Figure 7(b) shows two subdomains at level $k = 0$ are subdivided to get a mesh at level $k = 1$. Figure 7(c) shows two local tensor product meshes (corresponding to the two subdivided subdomains in Figure 7(b)) are subdivided into $(3,3)$-subdomains at level $k = 1$, and three subdomains at level $k = 1$ are further subdivided to get a mesh at level $k = 2$.

4.2 The dimension formulas

Now we are able to state our main result in the paper – the dimension formula for spline space $\overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}_{m,n})$, where $\mathscr{T}^{\varepsilon}_{m,n}$ is $\mathscr{T}_{m,n}$'s extended T-mesh associated with degree m, n.

Theorem 4.1. *Let V^+ be the number of crossing vertices of $\mathscr{T}^{\varepsilon}_{m,n}$. Assume $\mathscr{T}^{\varepsilon}_{m,n}$ has E_H interior horizontal l-edges and E_V interior vertical l-edges. Then*

$$\begin{aligned}&\dim \overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}_{m,n})\\&= (m-1)(n-1) + V^+ - (m-1)E_H - (n-1)E_V + \delta,\end{aligned} \tag{15}$$

where δ is the number of isolated subdomains of $\mathscr{T}^{\varepsilon}_{m,n}$ at all levels.

Before proving the theorem, we need some preparations.

Lemma 4.2. *Let $\mathscr{T}_{m,n}$ be a hierarchical T-mesh associated with (m,n), and $\mathscr{T}^{\varepsilon}_{m,n}$ be its extended T-mesh. $\mathbf{E}$ is the set of all the interior l-edges of $\mathscr{T}^{\varepsilon}_{m,n}$. Then the l-edges in $\mathbf{E}$ can be ordered as $E_t \prec E_{t-1} \prec \cdots \prec E_1$ such that the projection mapping*

$$\pi : \ W[\mathscr{T}_i] \longrightarrow W[E_i]$$

is surjective, $i = 1, 2, \ldots, t$. Here $\mathscr{T}_i$ is the T-mesh obtained by deleting $\{E_1, \cdots, E_{i-1}\}$ from $\mathscr{T}^{\varepsilon}_{m,n}$, and $E_i \subset \mathscr{T}_i$ is regarded as a l-edge in $\mathscr{T}_i$.

Proof. See the appendix. □

Base on the above lemma, we can decompose the conformality vector space $W[\mathscr{T}^{\varepsilon}_{m,n}]$ into direct sums of $W[E_i]$.

Theorem 4.3. *Let the notations be the same as in Lemma 4.2. Then*

$$W[\mathscr{T}_i] \cong W[\mathscr{T}_{i+1}] \oplus W[E_i], \quad i = 1, 2, \ldots, t.$$

Proof. Since $\mathscr{T}_i = \mathscr{T}_{i+1} \cup E_i$, $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}_{i+1}) \subset \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}_i)$. Correspondingly, $W[\mathscr{T}_{i+1}]$ can be regarded as a subspace of $W[\mathscr{T}_i]$ by taking every vector $\mathbf{k} \in W[\mathscr{T}_{i+1}]$ as a vector in $W[\mathscr{T}_i]$ whose components corresponding to the vertices of E_i are zero, and the remaining components are the same as $\mathbf{k}'s$.

Now consider the sequence

$$0 \longrightarrow W[\mathscr{T}_{i+1}] \xrightarrow{\quad i \quad} W[\mathscr{T}_i] \xrightarrow{\quad \pi \quad} W[E_i] \longrightarrow 0\ , \tag{16}$$

where i in the natural embedded mapping, and π is the projection mapping defined in Lemma 4.2. We will show that the sequence is exact. Since i is injective and π is surjective by Lemma 4.2, it is enough to show that $Im(i) = Ker(\pi)$.

On one hand, for any $\mathbf{k} \in W[\mathscr{T}_{i+1}]$, $i(\mathbf{k}) = \mathbf{k}'$ is a vector in $W[\mathscr{T}_i]$ whose components (conformality factors) corresponding to the vertices of E_i are zero. Thus $\pi(i(\mathbf{k})) = \mathbf{0} \in W[E_i]$, that is, $Im(i) \subset Ker(\pi)$.

On the other hand, for any $\mathbf{k} \in Ker(\pi)$, $\pi(\mathbf{k}) = \mathbf{0}$, that is, the components of $\mathbf{k}$ corresponding to the vertices of E_i are zero. Therefore $\mathbf{k} \in Im(i)$, i.e., $Ker(\pi) \subset Im(i)$.

Thus the sequence (16) is exact. Since $W[\mathscr{T}_{i+1}]$, $W[\mathscr{T}_i]$ and $W[E_i]$ are linear spaces, the sequence is split, and therefore $W[\mathscr{T}_i] \cong W[\mathscr{T}_{i+1}] \oplus W[E_i]$, as required. □

Based on the above theorem, we have

Corollary 4.4. *The set of functions $\cup_{i=1}^{t}\{B_j^i(x, y)\}_{j=1}^{j=r_i-m-1}$ defined in the proof of Lemma 4.2 form a basis for the spline space $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}_{m,n}^{\varepsilon})$.*

Now we are ready to prove the dimension formula (15).

Proof of Theorem 4.1. By Theorem 4.3, $W[\mathscr{T}_{m,n}^{\varepsilon}]$ can be decomposed into direct sums of $W[E_i]$:

$$W[\mathscr{T}_{m,n}^{\varepsilon}] \cong \oplus_{i=1}^{t} W[E_i].$$

The interior l-edges $\mathbf{E} = \{E_1, E_2, \ldots, E_t\}$ of $\mathscr{T}_{m,n}^{\varepsilon}$ can be divided into disjoint sets $\mathbf{E}^i$, $i = 0, 1, \ldots, l$, i.e., $\mathbf{E} = \cup_{i=0}^{l}\mathbf{E}^i$ and $\mathbf{E}^i \cap \mathbf{E}^j = \emptyset$, $i \neq j$. $\mathbf{E}^i$ is the set of interior l-edges of $\mathscr{T}_{m,n}^{\varepsilon}$ defined at level i, $i = 0, 1, \ldots, l$. In particular, $\mathbf{E}^0$ is the set of the interior l-edges of the initial extended tensor product mesh $\mathscr{T}_{\otimes}^{\varepsilon}$. By the order defined in Lemma 4.2, any element in $\mathbf{E}^i$ precedes any element in $\mathbf{E}^j$ for $i > j$.

By the direct sum decomposition of $W[\mathscr{T}^{\varepsilon}_{m,n}]$,

$$\dim W[\mathscr{T}^{\varepsilon}_{m,n}] = \sum_{i=0}^{l} \sum_{E_j \in \mathbf{E}^i} \dim W[E_j]. \tag{17}$$

Now we count $\dim W[E_j]$ according to the level of E_j from the highest level l to the lowest level 0.

By Lemma 3.2 and (14),

$$\dim W[E_j] = \dim \overline{\mathbf{S}}(m, m-1, E_j) = (v_j^+ - m + 1)_+$$

if E_j is a horizontal l-edge, and

$$\dim W[E_j] = \dim \overline{\mathbf{S}}(n, n-1, E_j) = (v_j^+ - n + 1)_+$$

if E_j is a vertical l-edge. Here v_j^+ is the number of crossing vertices of E_j in the mesh $\mathscr{T}_j$.

Consider all the interior l-edges $\mathbf{E}^i$ in a fixed level i $(i = l, l-1, \ldots, 0)$. $\mathbf{E}^i$ is obtained by subdividing some subdomains at level $i-1$ (except for $i = 0$ where $\mathbf{E}^0$ is the initial extended tensor product mesh). As shown in the proof of Lemma 4.2, if there is no isolated subdomain at level $i-1$, then one can order the l-edges in $\mathbf{E}^i$ such that

$$v_j^+ \geq \begin{cases} m-1, & \text{if } E_j \text{ is a horizontal l-edge} \\ n-1, & \text{if } E_j \text{ is a vertical l-edge} \end{cases}$$

If a subdomain $\mathscr{S}$ is isolated, then one can order the l-edges in the subdomain $\mathscr{S}$ such that the above equality holds for all but one vertical l-edge in $\mathscr{S}$. For the exceptional l-edge,

$$v_j^+ = n - 2.$$

Figure 9, illustrates an example for the order of l-edges in $\mathbf{E}^2$ in the case of $m = 4, n = 3$.

From the above facts, one has

$$\sum_{E_j \in \mathbf{E}^i} \dim W[E_j] = \sum_{E_j \in \mathbf{E}^i_h} (v_j^+ - m + 1) + \sum_{E_j \in \mathbf{E}^i_v} (v_j^+ - n + 1) + \delta_i, \quad i \geq 1, \tag{18}$$

where $\mathbf{E}^i_h$ is the set of horizontal l-edges in $\mathbf{E}^i$, $\mathbf{E}^i_v$ is the set of vertical l-edges in $\mathbf{E}^i$, and δ_i is the number of isolated subdomains at level $i-1$. For $i = 0$, since $\mathbf{E}^0$ is the set of l-edges of a tensor product mesh $\mathscr{T}^{\varepsilon}_{\otimes}$, it is direct to check that

$$\sum_{E_j \in \mathbf{E}^0} \dim W[E_j] = \sum_{E_j \in \mathbf{E}^0_h} (v_j^+ - m + 1) + \sum_{E_j \in \mathbf{E}^0_v} (v_j^+ - n + 1) + (m-1)(n-1). \tag{19}$$

Now the dimension formula (15) follows from (13), (17), (18) and (19). $\square$

4.3 Examples

In this subsection, we give two examples to count the dimensions of the spline spaces $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon}_{m,n})$.

Example 4.1. *In the case of $m = 2, n = 2$, $\mathscr{T}_{m,n}$ is a regular hierarchical T-mesh and we denote it as $\mathscr{T}$. By the dimension formula* (15),

$$\begin{aligned}\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) &= \dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^{\varepsilon}) \\ &= V^{+} - E + \delta + 1\end{aligned}$$

where, V^{+} is the number of crossing vertices, E is the number of interior l-edges, and δ is the number of isolated cells of $\mathscr{T}^{\varepsilon}$. This formula has been presented in [7].

Next we give a concrete example for the case $m = 3, n = 3$.

Example 4.2. *In this example, the hierarchical T-mesh $\mathscr{T}_{3,3}$ (which is the same as the T-mesh in Figure 7(c)) and its extension $\mathscr{T}^{\varepsilon}_{3,3}$ are illustrated in Figure 8. There*

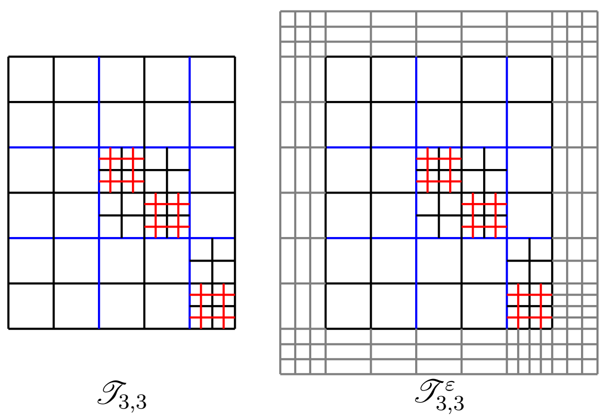


Figure 8: A hierarchical T-mesh $\mathscr{T}_{3,3}$ and its extension.

are three isolated subdomains (one at level 0 and two at level 1) in the middle of $\mathscr{T}^{\varepsilon}_{3,3}$. Thus $\delta = 3$. It is easy to count that $V^{+} = 166$, $E_H = 21$ and $E_V = 19$. So

$$\begin{aligned}&\dim \overline{\mathbf{S}}(3, 2, 2, 2, \mathscr{T}^{\varepsilon}_{3,3}) \\ &= V^{+} - (3-1)E_H - (3-1)E_V + \delta + (3-1)(3-1) \\ &= 93.\end{aligned}$$

We can also get the dimension by counting the number of basis functions.

There are $(12-4)(13-4) = 72$ basis functions over the 11×12 tensor product mesh $\mathscr{T}^{\varepsilon}_{3,3}$. At level 1, 7 basis functions are added to the basis functions set by subdividing two subdomains. At level 2, 14 basis functions are added to the basis functions set by subdividing three subdomains. Totally, there are $72 + 7 + 14 = 93$ basis functions, and therefore the dimension of $\overline{\mathbf{S}}(3, 2, 2, 2, \mathscr{T}^{\varepsilon}_{3,3})$ is 93.

5 Conclusions and future work

This paper presents a dimension formula for the spline space $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$ over certain type of hierarchical T-meshes. By using the smoothing co-factor method, we set up a bijection between the spline space $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$ and a conformality vector space $W[\mathscr{T}]$. Then we decompose $W[\mathscr{T}]$ into direct sum of simple linear spaces by using homological technique, and a dimension formula is thus obtained. At a by-product, we also obtain a set of basis functions for the spline space $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$.

It is important to construct a set of proper basis functions which have nice properties for the spline space in applications. We will study this topic in the future.

6 Appendix

In this section, the proof of Lemma 4.2 is given.

Proof. We first outline the proof strategy. For getting the result, we choose a set of bases $\{\mathbf{k}_j^i\}$ for $W[E_i]$ and construct a function $B_j^i(x,y) \in \overline{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}_i)(\cong W[\mathscr{T}_i])$ such that the conformality vector of $B_j^i(x,y)$ corresponding to the vertices of E_i is $\mathbf{k}_j^i$. Then $\pi: \ W[\mathscr{T}_i] \longrightarrow W[E_i]$ is surjective.

Before we discuss the construction of $B_j^i(x,y)$, some preparations are given as follows.

Preparations

1. Choose a set of bases for $W[E_i]$. If E_i is horizontal, let $\Delta_x: -\infty < x_1 < x_2 < \cdots < x_{r_i} < \infty$ be the x-coordinates of the vertices of E_i on $\mathscr{T}_i$. The B-spline basis functions $\{N[x_j, x_{j+1}, \cdots, x_{j+m+1}]\}_{j=1}^{r_i-m-1}$ is a set of bases of $\overline{\mathbf{S}}(m,m-1,\Delta_x)$ and $\{\mathbf{k}_j^i\}_{j=1}^{r_i-m-1}$ is a set of bases of $W[E_i]$, where $\mathbf{k}_j^i$ is the conformality vector of $N[x_j, x_{j+1}, \cdots, x_{j+m+1}]$. For a vertical interior l-edge, we can choose a set of bases of $W[E_i]$ similarly.

2. Define an operator on T-meshes. The operator $\mathbf{D}$ on a T-mesh $\mathscr{T}$ associated with an l-edge E_i is defined as follows. $\mathbf{D}_{E_i}(\mathscr{T}) = (E_i, \mathscr{T}_1)$, where $\mathscr{T}_1$ is the T-mesh by removing E_i from $\mathscr{T}$. Then $\pi_1 \circ \mathbf{D}_{E_i}(\mathscr{T}) = E_i$, and $\pi_2 \circ \mathbf{D}_{E_i}(\mathscr{T}) = \mathscr{T}_1$.

3. Define an order of $\mathbf{E}$. The interior l-edges $\mathbf{E}$ of $\mathscr{T}_{m,n}^{\varepsilon}$ can be divided into disjoint sets $\mathbf{E}^i$, $i = 0, 1, \ldots, l$, i.e., $\mathbf{E} = \cup_{i=0}^{l}\mathbf{E}^i$ with $\mathbf{E}^i \cap \mathbf{E}^j = \emptyset$, $i \neq j$. $\mathbf{E}^i$ is the set of interior l-edges of $\mathscr{T}_{m,n}^{\varepsilon}$ defined at level i, $i = 0, 1, \ldots, l$, where l is the level of $\mathscr{T}_{m,n}$. The order "$\prec$" between $\mathbf{E}^p$ and $\mathbf{E}^q$ is defined as: $\forall e^p \in \mathbf{E}^p, \forall e^q \in \mathbf{E}^q$, if $q < p$, then $e^q \prec e^p$. In the following, the order "$\prec$" inside $\mathbf{E}^p$ is presented. Consider $\mathbf{E}^p$, and suppose the T-mesh $\widetilde{\mathscr{T}^p}$ is obtained by deleting the l-edges

$\cup_{i=p+1}^{l}\mathbf{E}^i$ from $\mathscr{T}_{m,n}^{\varepsilon}$. Without loss of generality, it is enough to consider the case of $p = l$.

Then, the order inside $\mathbf{E}^l$ is defined as follows with respect to the case of $l = 0$ and $l > 0$, respectively.

- $l = 0$. The T-mesh $\mathscr{T}$ is a tensor product mesh $\mathscr{T}_{\otimes}$. The progress of choosing interior l-edges sequence is presented in the following.

  $\mathscr{T} \leftarrow \mathscr{T}_{\otimes}$
  $S_E \leftarrow$ the set of interior l-edges of $\mathscr{T}$
  $i \leftarrow 1$
  while $S_E \neq \emptyset$ **do**
    if there is no trivial interior l-edge **then**
      choose any interior l-edge E_i^0
    else
      choose a trivial interior l-edge E_i^0 (i.e $W[E_i^0] = 0$)
    end if
    $\mathscr{T} \leftarrow \pi_2 \circ \mathbf{D}_{E_i^0}(\mathscr{T})$
    $S_E \leftarrow$ the set of interior l-edges of $\mathscr{T}$
    $i \leftarrow i + 1$
  end while

  By this progress, we get a sequence of $\mathbf{E}^0$ denoted as $E_1^0, E_2^0, \cdots$, where E_i^0 is chosen earlier than E_{i+1}^0. The order is defined as $E_{i+1}^0 \prec E_i^0$

- $l > 0$. Select an l-edge $E \in \mathbf{E}^l$. Then E must intersect with an (m, n)-subdomain D at level $(l-1)$. If E is vertical, we consider all the vertical l-edges in $\mathbf{E}^l$ which intersect with D and arrange them from left to right. Then E is the i-th l-edge. By the structure of $\mathscr{T}_{m,n}^{\varepsilon}$, E is also the i-th l-edge in another (m, n)-subdomain at level $(l-1)$ which intersects with E. Hence we can denote the position of E as $L(E) = i$. Considering the local tensor product structure on D at level $(l-1)$, $L(E) = i < m$. If E is horizontal, $L(E)$ can be defined in a similar way by arranging horizontal l-edges in $\mathbf{E}^l$ from top to bottom and $L(E) < n$.
  Define l-edge sets A_1, A_2, A_3, A_4, A_5 as

  $$\begin{aligned} A_1 =& \{E \in \mathbf{E}^l : \text{ If } E \text{ is vertical, } L(E) < m-2; \\ & \text{If } E \text{ is horizontal, } L(E) < n-2\}, \\ A_2 =& \{E \in \mathbf{E}^l : E \text{ is horizontal, } L(E) = n-2\}. \\ A_3 =& \{E \in \mathbf{E}^l : E \text{ is vertical, } L(E) = m-2\}, \\ A_4 =& \{E \in \mathbf{E}^l : E \text{ is horizontal, } L(E) = n-1\}, \\ A_5 =& \{E \in \mathbf{E}^l : E \text{ is vertical, } L(E) = m-1\}. \end{aligned}$$

  The progress of choosing interior l-edge sequence is presented in the following.

$\mathscr{T}^0 \leftarrow \mathscr{T}^{\varepsilon}_{m,n}$
for $i = 1$ to 5 **do**
  $\widetilde{\mathscr{T}} \leftarrow \mathscr{T}^{i-1}$
  $S_E \leftarrow$ the set of interior l-edges of $\widetilde{\mathscr{T}}$ in A_i
  $j \leftarrow 1$
  while $S_E \neq \emptyset$ **do**
    if there is no trivial l-edge in S_E **then**
      choose any interior l-edge $E^l_{i,j}$ from S_E
    else
      choose any trivial l-edge $E^l_{i,j}$ from S_E
    end if
    $\widetilde{\mathscr{T}} \leftarrow \pi_2 \circ \mathbf{D}_{E^l_{i,j}}(\widetilde{\mathscr{T}})$
    $S_E \leftarrow$ the set of interior l-edges of $\widetilde{\mathscr{T}}$ in A_i
    $j \leftarrow j + 1$
  end while
  $\mathscr{T}^i \leftarrow \widetilde{\mathscr{T}}$
end for

This progress generates an interior l-edge sequence $E^l_{1,1}$, $E^l_{1,2}, \cdots$, E^l_{1,j_1}, $E^l_{2,1}$, $E^l_{2,2}$, $\cdots$, E^l_{2,j_2}, $\cdots$, E^l_{5,j_5}. If one l-edge e_1 is chosen earlier than another l-edge e_2 in the sequence, we define $e_2 \prec e_1$.

For an example, see Figure 9, we consider the case of $l = 2$. Here $m = 4, n = 3, p = 6, q = 6$. An l-edge in the T-mesh belongs to the set in the right side of Figure 9 whose name has the same color as the l-edge. And its order of an interior l-edge sequence of $\mathbf{E}^2$ is labeled near itself.

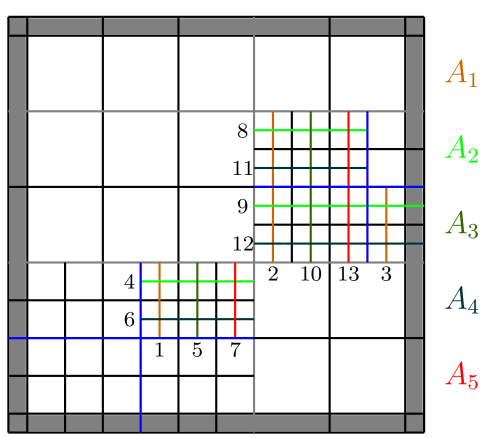


Figure 9: An example of $\mathbf{E}^2$, where $m = 4$, $n = 3$, $p = 6$, $q = 6$.

$B^i_j(x, y)$ **construction**

Now we are ready to construct $B_j^i(x, y)$.

When $l > 0$, the B-spline function $N_j^i(x, y)$ can be constructed by the tensor product mesh over (m, n)-subdomains at level $(l-1)$ contained e. Here $N_j^i(x, y)$ is related to the knots of an interior l-edge e in the sequence associated with A_1(or A_2, A_3). It is a tensor-product B-spline function, which equals to $B_j^i(x, y)$ up to a nonzero constant. For trivial interior l-edges, no function is introduced. Suppose that we construct the functions associated with all the l-edges whose orders are "larger" than the order of E in the sequence associated with A_4. We now construct the functions associated with E. E is a non-trivial horizontal l-edges and we denote the x-coordinates of E as $\{x_1, x_2, \cdots, x_d\}$. Let α be the number of vertices formed by two l-edges with level l intersection. This type of vertex is called an (l, l)-vertex later. The i-th knots sequence of E's x-coordinates is $\{x_i, \cdots, x_{i+(m+1)}\}$. According to the structure of the current mesh, $\alpha \in \{0, 1, 2\}$.

1. $\alpha = 0$. If $l \geq 2$, a B-spline can be constructed over a tensor product mesh over all the (m, n)-subdomains at level $(l-2)$ containing E's i-th knot sequence. If $l = 1$, we can construct a B-spline according to the tensor product mesh of level 0. The required function $B_j^i(x, y)$ is the same as the B-spline function up to a nonzero constant.

2. $\alpha = 1$. If $l \geq 2$, consider the unique $(l, l)-$vertex, denoted as Q. Assume the (m, n)-subdomain which contains Q at level $(l-2)$ is Σ. If the interior vertical l-edge that contains Q goes through Σ, we can construct a B-spline similar to the case of $\alpha = 0$. If it does not go through Σ, one of its endpoint must be within Σ and another is outside of Σ according to the structure of the special hierarchical T-mesh. Suppose the intersection point between the vertical l-edge and Σ is R. We extend the l-edge from the endpoint within Σ and cut the boundary of Σ at P. Then the $n+2$ vertices can be used to construct a B-spline $N_1(x, y)$ by a tensor product mesh over all (m, n)-subdomains containing the given $(m+2)$ vertices. These $n+2$ vertices consist of P, R, and n vertices in the original vertical l-edge within Σ. Therefore, the conformality vector of $N_1(x, y)$ is nontrivial at these $(m+2)$ vertices and trivial at the new vertices without P in the process of extending. Suppose k_1 is the conformality factor at P. If another function $g(x, y)$ is constructed such that its conformality factor at P is $-k_1$ and the conformality factors at the new vertices without P and the given $(m+2)$ vertices are zeros, then $f(x, y) = N_1(x, y) + g(x, y)$ differs by a multiplier λ $(\neq 0)$ with the function $B_i^j(x, y)$ which we need by the linear property of $\mathscr{K}$ and Example 3.2.

   Now we discuss the construction of $g(x, y)$. If we remove Q from the set consisting of P and all the vertices in the original vertical l-edge, the number of remaining vertices is not less than $(n+2)$. We choose $(n+2)-$vertices from these remaining vertices and P must be one of them. Similar to the case of $\alpha = 0$, there is no (l, l)-vertex in these $(n+2)-$vertices. So, a B-spline $N_2(x, y)$

is constructed and we denote k_2 as the conformality factor of $N_2(x, y)$ at P. By Example 3.2, $k_2 \neq 0$ and the conformality vector of $N_2(x, y)$ is trivial at the new vertices without P and those given $(m + 2)$ vertices according to Q removed. Then we define $g(x, y) = -k_1/k_2 N_2(x, y)$.

For an example, see Figure 10, where $m = 3$, $n = 3$. Here P is shown as "□" in the figure. $N_1(x, y)$ is constructed over the tensor product mesh shown in the upper right of this figure and the one in the lower right is used to construct $N_2(x, y)$. Compute k_1 of $N_1(x, y)$ and k_2 of $N_2(x, y)$ at P as we have shown in Examples 3.1 and 3.2. Then $f(x, y)$ is given by

$$f(x, y) = N_1(x, y) - \frac{k_1}{k_2} N_2(x, y). \tag{20}$$

When $l = 1$, a B-spline associated with the given $m + 2$ vertices can be

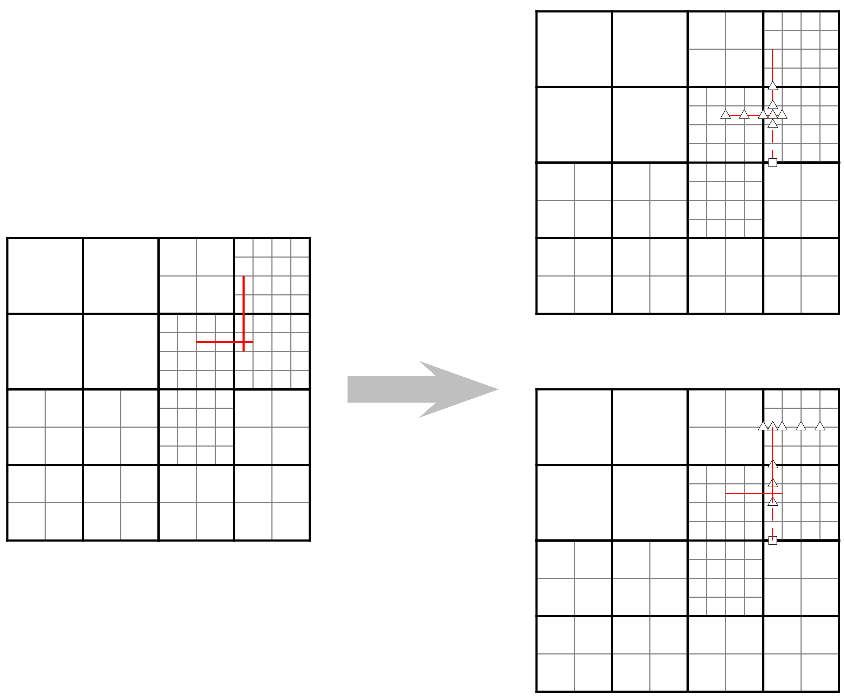


Figure 10: Construction of the linear combination of B-splines in the case of $m = n = 3$.

constructed by the tensor product mesh of level 0.

3. $\alpha = 2$. If $l \geq 2$, there are two $(l, l)-$vertices P_1 and P_2 in the given $(m + 2)$ vertices. One of the two vertical l-edges at P_1, P_2 respectively must go through Σ' that occupied by (m, n)-subdomains containing P_1 and P_2 at level $(l - 2)$. So, we can construct the function by the method presented in the case of $\alpha = 1$. When $l = 1$, the function can be constructed by the tensor product mesh of level 0 and the case $l \geq 2$.

When all the l-edges in the sequence associated with A_4 are deleted, there are no $(l,l)-$vertices at the l-edges in A_5. So we can construct the function we need by the method presented in the case of $\alpha = 0$.

When $l = 0$, by the order defined on $\mathbf{E}^0$ and the tensor product mesh at level 0, B-spline surfaces can always be constructed. So as stated previously, π is surjective.

□

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References

[1] T. Sederberg, J. Zheng, A. Bakenov, and A. Nasri, "T-splines and T-NURCCs.," *ACM Transactions on Graphics*, vol. 22, no. 3, pp. 161–172, 2003.

[2] T. Sederberg, D. Cardon, G. Finnigan, N. North, J. Zheng, and T. Lyche, "T-spline simplification and local refinement," *ACM Transactions on Graphics*, vol. 23, no. 3, pp. 276–283, 2004.

[3] J. Deng, F. Chen, and Y. Feng, "Dimensions of spline spaces over T-meshes," 2006.10.

[4] J. Deng, F. Chen, X. Li, C. Hu, W. Tong, Z. Yang, and Y. Feng, "Polynomial splines over hierarchical T-meshes," *Graphical Models*, vol. 74, no. 4, pp. 76–86, 2008.

[5] S. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*. Springer, 2002.

[6] C. Li, R. Wang, and F. Zhang, "Improvement on the dimension of spline spaces on T-mesh," *Journal of Information and Computational Science*, vol. 3, no. 2, pp. 235–244, 2006.

[7] J. Deng, F. Chen, and L. Jin, "Dimensions of biquadratic spline spaces over T-meshes," *Preprint, arXiv:math.NA/0804.2533*, 2008.

[8] B. Mourrain, "On the dimension of spline spaces on planar T-subdivisions," *Preprint,inria-00533187,version 1-5 Nov 2010*, 2010.

[9] X. Li and F. Chen, "On the instability in the dimension of splines space over T-meshes," *Preprint*, 2011.

[10] R. Wang, "On the analysis of multivariate splines in the case of arbitrary partition," *Scientia Sinica*, 1979.

[11] L. Billera, "Homology of smooth splines: generic triangulations and a conjecture of strang," *Transactions of the American Mathematical Society*, vol. 310, no. 1, 1988.